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SOLUTION OF A PROBLEM IN THE CONTROL OF THE MOTION OF NONLINEAR SYSTEMS

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ABSTRACT

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An approximate method is given for solving a problem in the control of the motion of nonlinear systems. The convergence of the successive approximations is demonstrated. A procedure is indicated for implementing the solution of this problem on electronic analog equipment. As an application of the method, the problem of the accelerated drive of an Anschütz gyrocompass in the meridian is solved when the latter instrument has a nonlinear restoring force.

Author

1. YA. N. ROYTENBERG'S "ZEROING-IN" PROBLEM IN NONLINEAR SYSTEMS (REFERENCE 1)

Let the motion of a certain controlled system be described by the following type of differential equations:

$$\frac{dx_\nu}{dt} = \sum_{\mu=1}^{n} a_{\nu\mu}(t)x_\mu + \varepsilon\Psi_\nu(x_1, x_2, \ldots, x_n) + f_\nu(t) + q_\nu(t) \quad (\nu = 1, 2, \ldots, n). \tag{1.1}$$

Here x_ν are the phase coordinates of the investigated system, $f_\nu(t)$ are prescribed external forces, representing continuous functions of the time, $q_\nu(t)$ are additional external forces (control forces), whose law of variation is yet to be determined, $a_{\nu\mu}(t)$ are known continuous functions of the time, ϵ is some positive parameter.

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The system of scalar functions (1.1) is equivalent to the matrix equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \varepsilon\Psi(\mathbf{x}) + \mathbf{f}(t) + \mathbf{q}(t), \tag{1.2}$$

where $\mathbf{x}$, $\mathbf{A}(t)$, $\Psi(x)$, $\mathbf{f}(t)$, $\mathbf{q}(t)$[1] denote the following matrices:

$$\mathbf{x} = \| x_\nu \|, \quad \mathbf{A}(t) = \| a_{\nu\mu}(t) \|, \quad \Psi(\mathbf{x}) = \| \Psi_\nu(x_1, x_2, \ldots, x_n) \|,$$
$$\mathbf{f}(t) = \| f_\nu(t) \|, \quad \mathbf{q}(t) = \| q_\nu(t) \| \quad (\nu, \mu = 1, 2, \ldots, n).$$

Using the symbol $\Theta(t)$ to denote the fundamental matrix for the matrix differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}, \tag{1.3}$$

we can transform from the nonlinear matrix differential equation (1.2) to the nonlinear matrix integral equation

$$\mathbf{x}(t) = \mathbf{N}(t, t_0)\mathbf{x}(t_0) + \varepsilon \int_{t_0}^{t} \mathbf{N}(t, \sigma)\Psi(\mathbf{x}(\sigma))\,d\sigma + \int_{t_0}^{t} \mathbf{N}(t, \sigma)[\mathbf{f}(\sigma) + \mathbf{q}(\sigma)]\,d\sigma, \tag{1.4}$$

where

$$\mathbf{N}(t, \sigma) = \Theta(t)\Theta^{-1}(\sigma) \tag{1.5}$$

is a matrix weighting function for the matrix differential equation (1.3). /166

$\Theta^{-1}(\sigma)$ denotes in equation (1.5) the inverse matrix of $\Theta(\sigma)$.

Confining our treatment to the case of control forces that remain constant in the time interval $t_0 \leqslant t \leqslant t_1$,

$$q_\nu(t) = q_\nu^* \quad (\nu = 1, 2, \ldots, n), \tag{1.6}$$

we pose the problem of finding the constant vector $q^* = \| q_\nu^* \|$, for which the condition

$$\mathbf{x}(t_1) = \varkappa \qquad (1.7)$$

is satisfied, where $\varkappa$ is a prespecified constant vector, t_1 is a predetermined instant of time.

To solve the problem as stated, we direct our attention to equation (1.4), according to which the condition (1.7) will be satisfied if the vector $\mathbf{q}^*$ is chosen such that the following relation is fulfilled:

$$\mathbf{q}^* = \boldsymbol{\gamma} - \varepsilon \mathbf{W}^{-1}(t_1) \int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma) \Psi(\mathbf{x}(\sigma))\, d\sigma,$$

$$\mathbf{x}(t) = \mathbf{s}(t) + \mathbf{W}(t)\mathbf{q}^* + \varepsilon \int_{t_0}^{t} \mathbf{N}(t, \sigma) \Psi(\mathbf{x}(\sigma))\, d\sigma, \quad t_0 \leqslant t \leqslant t_1, \qquad (1.8)\text{-}(1.9)$$

where $\boldsymbol{\gamma}$, $\mathbf{s}(t)$, $\mathbf{W}(t)$ denote the matrices

$$\boldsymbol{\gamma} = \mathbf{W}^{-1}(t_1)\,[\varkappa - \mathbf{s}(t_1)],$$

$$\mathbf{s}(t) = \mathbf{N}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{N}(t, \sigma)\mathbf{f}(\sigma)\, d\sigma, \qquad (1.10)\text{-}(1.12)$$

$$\mathbf{W}(t) = \int_{t_0}^{t} \mathbf{N}(t, \sigma)\, d\sigma.$$

To determine the vector $\mathbf{q}^*$ from the relations (1.8) and (1.9), we apply the method of successive approximations. For the zeroth approximation, we assume

$$\mathbf{q}^0 = \boldsymbol{\gamma}. \qquad (1.13)$$

The solution of the nonlinear matrix integral equation (1.9) for $\mathbf{q}^* = \boldsymbol{\gamma}$ or, what is equivalent, the differential equation (1.2) for $\mathbf{q}(t) = \boldsymbol{\gamma}$, will be symbolized by $\mathbf{x}^0(t)$.

Substituting $\mathbf{x}^0(t)$ into the right-hand side of (1.8) and letting

$$\mathbf{x}^0(t) = \mathbf{x}^0(\sigma_j), \quad t_0 \leqslant \sigma_{j-1} \leqslant t \leqslant \sigma_j \leqslant t_1$$
$$(j = 1, 2, \ldots, m, \quad \sigma_0 = t_0, \quad \sigma_m = t_1), \tag{1.14}$$

we obtain the following expression for the control force vector, which we will adopt as the first approximation, to wit:

$$\mathbf{q}^1 = \boldsymbol{\gamma} - \sum_{j=1}^{m} \varepsilon \mathbf{V}^j \boldsymbol{\Psi}(\mathbf{x}^0(\sigma_j)), \tag{1.15}$$

where

$$\mathbf{V}^j = \mathbf{W}^{-1}(t_1)\mathbf{W}^j, \quad \mathbf{W}^j = \int_{\sigma_{j-1}}^{\sigma_j} \mathbf{N}(t_1, \sigma)\, d\sigma \qquad (j = 1, 2, \ldots, m), \tag{1.16}$$

We define the $(k + 1)$th approximation for the control force vector by the relation /167

$$\mathbf{q}^{k+1} = \boldsymbol{\gamma} - \sum_{j=1}^{m} \varepsilon \mathbf{V}^j \boldsymbol{\Psi}(\mathbf{x}^k(\sigma_j)) \qquad (k = 0, 1, 2, \ldots),$$

where

$$\mathbf{x}^k(\sigma_j) = \mathbf{x}^k(t)\big|_{t=\sigma_j} \qquad (j = 1, 2, \ldots, m), \text{ a } \mathbf{x}^k(t) \tag{1.17}$$

is, in turn, the solution of the matrix differential equation (1.2) for $\mathbf{q}(t) = \mathbf{q}^k$.

Consequently, to obtain the successive approximations, the nonlinear system (1.2) is first simulated for the prescribed initial conditions and for $\mathbf{q}(t) = \boldsymbol{\gamma}$. The generation of the vectors $\mathbf{x}^0(\sigma_j)$ $(j = 1, 2, \ldots, m)$ is ensured in the course of integration. Substitution of these vectors into the right-hand side of equation (1.15) yields the first approximation $\mathbf{q}^1$. The subsequent approximations are obtained analogously. It is apparent from (1.10) and (1.16) that in order to determine the matrices γ, V^j, the matrix weighting function $N(t_1,\sigma)$ must be known. We are aware (ref. 2) that

$$N(t_1, \sigma) = Z^{\tau}(\sigma). \tag{1.18}$$

Here $Z(\sigma)$ is the fundamental matrix of the conjugate system

$$\frac{dz}{d\sigma} = -A^{\tau}(\sigma)\,z; \tag{1.19}$$

satisfying the condition

$$Z(t_1) = E_n, \tag{1.20}$$

where E_n is a unit matrix of nth rank. The symbols $Z(\sigma)$ and $A(\sigma)$ denote the transposes of the matrices $Z(\sigma)$ and $A(\sigma)$.

Inasmuch as we only know, at this point, the final values for the variables of the conjugate system, in order to determine the matrix weighting function $N(t_1,\sigma)$ by means of electronic analog devices it is first of all necessary to determine the initial values for the variables of the conjugate system so as to meet the condition (1.20). These can be found by integrating "back" the system (1.19) (see, e.g., ref. 2, 3).

For the case when the fundamental matrix of the controlled system is constant, we have (ref. 4)

$$N(t_1, \sigma) = X(t_1 - \sigma), \tag{1.21}$$

where $X(t)$ is the fundamental matrix of the matrix differential equation (1.3) satisfying the condition

$$X(t_0) = E_n. \tag{1.22}$$

Hence it is clear that the matrices γ, V^j in this special case can be computed without resorting to the conjugate system.

2. PROBLEM OF THE ACCELERATED MERIDIANAL DRIVE OF THE ANSCHÜTZ GYROCOMPASS

As an application of the method outlined above, we will solve the problem of an accelerated drive imparted to the Anschütz gyrocompass in the meridian when this instrument has a nonlinear restoring force.

The equations of precessional motion of the Anschütz gyrocompass, which absorbs natural oscillations by means of air jets, have the form (ref. 5)

$$-H\dot{\alpha} + lP\beta = HU\sin\varphi + Q(t), \quad H\dot{\beta} + (HU\cos\varphi)\alpha + N\beta = 0. \tag{2.1}$$

Here α is the azimuthal angle of rotation of the gyrocompass, β is the angle of ascent of the southern tip of the gyro rotor above the horizontal plane. H denotes the kinetic moment of the gyro, lP is the static moment of the gyro, U is the angular velocity of diurnal rotation of the earth, ϕ is the latitude of the point of observation, N is the reactive torque due to air pressure, ° which is directed along the axis of the figure, and Q(t) is an additional generalized external force (representing the torque relative to the eastern axis of the gyro and applied to accelerate the gyrocompass in the meridian), whose law of variation is subject to determination. /168

Denoting

$$x_1 = \alpha + \frac{N\sin\varphi}{lP\cos\varphi}, \quad x_2 = \beta - \frac{HU\sin\varphi}{lP}, \tag{2.2}$$
$$k^2 = \frac{lPU\cos\varphi}{H}, \quad 2s = \frac{N}{H}, \quad q_1(t) = -\frac{Q(t)}{H}$$

and introducing the nonlinear term $\epsilon\psi_2(x_1) = -\epsilon x_1^3$, we write the equations of precessional motion of the Anschütz gyrocompass in the following form:

$$\begin{aligned} \dot{x}_1 &= \frac{k^2}{U\cos\varphi} x_2 + q_1(t), \\ \dot{x}_2 &= -U\cos\varphi\, x_1 - 2sx_2 + \varepsilon\Psi_2(x_1). \end{aligned} \tag{2.3}$$

The integral equations equivalent to the differential equations (2.3) have the form

$$\begin{aligned} x_\nu(t) = \sum_{\mu=1}^{2} X_{\nu\mu}(t)x_\mu(0) + \varepsilon\int_0^t X_{\nu 2}(t-\sigma)\Psi_2(x_1(\sigma))\,d\sigma + \\ + \int_0^t X_{\nu 1}(t-\sigma)\,q_1(\sigma)\,d\sigma \qquad (\nu = 1, 2), \end{aligned} \tag{2.4}$$

where $X_{\nu\mu}(t)$ are elements of the fundamental matrix of equations (2.3) for $q_1(t) \equiv 0$ and $\epsilon = 0$, satisfying the conditions

$$X_{\nu\mu}(0) = \begin{cases} 1 & \nu = \mu, \\ 0 & \nu \neq \mu, \end{cases} \qquad (\nu, \mu = 1, 2). \tag{2.5}$$

We now require that at the time $t = t_1$ the gyrocompass be driven in the meridian, in other words, that

$$x_1(t) = x_2(t_1) = 0. \tag{2.6}$$

Since there is only one controlling force and the number of controlled phase coordinates is equal to two, we pursue the basic idea of reference 1 and divide the interval $(0, t_1)$ into two equal subintervals, designating q_{11}^* and q_{12}^* as the values of the step function $q_1(t)$ on the indicated subintervals.

Then, on the basis of equation (2.4) we obtain the following relations for determining these quantities:

$$q_{1j}^{*} = \gamma_{1j} - \sum_{\nu=1}^{2} \varepsilon W_{j\nu}^{*} \int_{0}^{t_1} X_{\nu 2}(t_1 - \sigma)\, \Psi_2(x_1(\sigma))\, d\sigma \qquad (j = 1,\ 2). \tag{2.7}$$

Here

$$\gamma_{1j} = -\sum_{\nu=1}^{2} W_{j\nu}^{*} s_\nu, \qquad s_\nu = \sum_{\mu=1}^{2} X_{\nu\mu}(t_1)\, x_\mu(0) \qquad (\nu,\ j = 1,\ 2),$$

and $W_{j\nu}$ are elements of the inverse matrix of $W = \| W_{\nu 1}^{j} \|$ (2 x 2), where

$$W_{\nu 1}^{j} = \int_{\tau_{j-1}}^{\tau_j} X_{\nu 1}(t_1 - \sigma)\, d\sigma \quad \left(\nu, j = 1, 2, \quad \tau_0 = 0, \quad \tau_1 = \frac{t_1}{2}, \quad \tau_2 = t_1\right).$$ /169

This problem was solved with the aid of an electronic analog for the following values of the parameters:

$$\begin{aligned} k^2 &= 1.53921 \cdot 10^{-6}\ \text{sec}^{-1}, \\ U \cos\varphi &= 3.646 \cdot 10^{-5}\ \text{sec}^{-1}, \\ s &= 2.97756 \cdot 10^{-4}\ \text{sec}^{-1}, \\ \varepsilon &= 0.4 \cdot 10^{-4}\ \text{sec}^{-1}, \\ H &= 290.000\ \text{g-cm-sec}. \end{aligned} \tag{2.8}$$

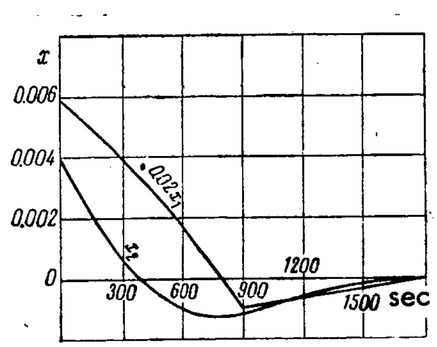

Figure 1

The drive time t_1 = 1800 sec. The initial deviations were

$$x_1(0) = 0.3, \qquad x_2(0) = 0.004. \tag{2.9}$$

The electronic analog solution yielded the following results:

$$\gamma_{11} = -0.3962\cdot 10^{-3} \text{ sec}^{-1}, \quad \gamma_{12} = -0.6024\cdot 10^{-4} \text{ sec}^{-1}, \tag{2.10}$$

which were taken as the zeroth approximation for the step-function force $q_1(t)$.

In simulating the system (2.3) in correspondence with the values of the parameters (2.8), with the initial conditions (2.9) and force (2.10), we obtain

$$x_{11}{}^0 = 0.2085,\ x_{12}{}^0 = 0.1146,\ x_{13}{}^0 = -0.0465,\ x_{14}{}^0 = -0.0378, \quad x_{15}{}^0 = -0.0228,\ x_{16}{}^0 = -0.0045, \tag{2.11}$$

where $x_{1j}{}^0$ are the values of the function $x_1{}^0(t)$ at the points $t = \sigma_j$ ($\sigma_j - \sigma_{j-1} = 300$ sec; $j = 1, 2, \ldots, 6$; $\sigma_0 = 0$; $\sigma_6 = t_1$).

Substituting $x_1{}^0(t)$ into the right-hand sides of (2.7) and letting

$$x_1{}^0(t) = x_{1j}{}^0, \quad \sigma_{j-1} \leqslant t \leqslant \sigma_j \quad (j = 1, 2, \ldots, 6), \tag{2.12}$$

we obtain the first approximation:

$$q_1^1(t) = \begin{cases} -0.3955\cdot 10^{-3}\ \text{сек}^{-1}, & 0 \leqslant t \leqslant 900 \text{ sec}, \\ 0.6350\cdot 10^{-3}\ \text{сек}^{-1}, & 900 \leqslant t \leqslant 1800 \text{ sec}. \end{cases} \tag{2.13}$$

With this force acting at the instant $t_1 = 1800$ sec, the coordinates x_1, x_2 go to zero. The error, normalized to a scale of 100 V, does not exceed 1%.

The process of driving the gyrocompass in the meridian by means of the force (2.13) is illustrated graphically by the functions x_1, x_2 in figure 1.

3. PROOF OF CONVERGENCE OF THE SUCCESSIVE APPROXIMATIONS

Let us rewrite equations (1.8) and (1.9) in the more general form (time appearing explicitly in the nonlinear terms)

$$\mathbf{q} = \gamma - \varepsilon \mathbf{W}^{-1}(t_1) \int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma)\, \Psi(\mathbf{x}(\sigma), \sigma)\, d\sigma,$$

$$\mathbf{x}(t) = \mathbf{s}(t) + \mathbf{W}(t)\,\mathbf{q} + \varepsilon \int_{t_0}^{t} \mathbf{N}(t, \sigma)\, \Psi(\mathbf{x}(\sigma), \sigma)\, d\sigma, \quad t_0 \leqslant t \leqslant t_1. \qquad (3.1)\text{-}(3.2)$$

According to the approximate method outlined in section 1, we have

$$\mathbf{q}^1 = \gamma - \varepsilon \mathbf{W}^{-1}(t_1) \int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma)\, \Psi(\mathbf{x}^0(\sigma), \sigma)\, d\sigma,$$

$$\mathbf{x}^0(t) = \mathbf{s}(t) + \mathbf{W}(t)\gamma + \varepsilon \int_{t_0}^{t} \mathbf{N}(t, \sigma)\, \Psi(\mathbf{x}^0(\sigma), \sigma)\, d\sigma, \quad t_0 \leqslant t \leqslant t_1, \qquad (3.3)\text{-}(3.4)$$

$$\mathbf{q}^{k+1} = \gamma - \varepsilon \mathbf{W}^{-1}(t_1) \int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma)\, \Psi(\mathbf{x}^k(\sigma), \sigma)\, d\sigma,$$

$$\mathbf{x}_k(t) = \mathbf{s}(t) + \mathbf{W}(t)\left\{\gamma - \varepsilon \mathbf{W}^{-1}(t_1) \int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma)\, \Psi(\mathbf{x}^{k-1}(\sigma), \sigma)\, d\sigma\right\} +$$

$$+ \varepsilon \int_{t_0}^{t} \mathbf{N}(t, \sigma)\, \Psi(\mathbf{x}^k(\sigma), \sigma)\, d\sigma, \quad t_0 \leqslant t \leqslant t_1 \quad (k = 1, 2, \ldots). \qquad (3.5)\text{-}(3.6)$$ /170

To prove the convergence of the successive approximations, we interject the following postulates:

1) The vector function $\psi(\mathbf{x}, t)$ is continuous with respect to all arguments in some closed region $\mathbf{D}$ $(n + 1)$-dimensional space $(\mathbf{x}, t)$, where

$$\mathbf{D} \supset \mathbf{D}_{2\Delta}{}^0(\mathbf{x}, t) = \{|\mathbf{x} - \mathbf{z}^0(t)| \leqslant 2\Delta,\ t_0 \leqslant t \leqslant t_1\},$$
$$\mathbf{z}^0(t) = \mathbf{s}(t) + \mathbf{W}(t)\gamma. \qquad (3.7)\text{-}(3.8)$$

2) In the region $\mathbf{D}$, the vector function $\psi(\mathbf{x}, t)$ satisfies the Lipshits r-condition. This means that for any two points (x'', t) and (x', t) of the region $\mathbf{D}$, the condition

$$|\Psi(\mathbf{x}'', t) - \Psi(\mathbf{x}', t)| \leqslant r|\mathbf{x}'' - \mathbf{x}'|, \quad r = \text{const}, \tag{3.9}$$

is satisfied, where $|\mathbf{x}|$ denotes the normal form of the vector $\mathbf{x}$.

3) The parameter ϵ is determined according to the condition

$$0 < \varepsilon < E_0^*, \quad E_0^* = \min\{\varepsilon_1 = \Delta / MK(t_1 - t_0), \ \varepsilon_2 = \Delta / NK(t_1 - t_0), \ \varepsilon_0\}, \tag{3.10}$$

where M, N, K indicate the following quantities:

$$M = \max_{t, \sigma \in [t_0, t_1]} |\mathbf{W}(t)\mathbf{W}^{-1}(t_1)\mathbf{N}(t_1, \sigma)|, \quad N = \max_{t, \sigma \in [t_0, t_1]} |\mathbf{N}(t, \sigma)|,$$
$$K = \sup_{(x, t) \in D} |\Psi(\mathbf{x}, t)|, \tag{3.11}$$

and the numerical value of ϵ_0 will be determined below.

We will show that under these assumptions, all approximations (3.6) for any $t \in [t_0, t_1]$ will fit completely within a tube $\mathbf{D}_2^0(\mathbf{x}, t) \subset \mathbf{D}$.

For this, we rewrite (3.4) in the form

$$\mathbf{x}^0(t) = \mathbf{z}^0(t) + \varepsilon \int_{t_0}^{t} \mathbf{N}(t, \sigma)\Psi(\mathbf{x}^0(\sigma), \sigma)\, d\sigma, \quad t_0 \leqslant t \leqslant t_1. \tag{3.12}$$

Applying the iterative method of Picard to the latter (see, e.g., ref 6), it is not too difficult to show that there exists for it a unique solution, which is defined and continuous on the time interval $t_0 \leq t \leq t_1$, fitting completely inside the tube $\mathbf{D}_\Delta^0(\mathbf{x}, t)$, i.e.,

$$\mathbf{x}^0(t) \in \mathbf{D}^0_\Delta(\mathbf{x}, t) = \{|\mathbf{x} - \mathbf{z}^0(t)| \leqslant \Delta, \ t_0 \leqslant t \leqslant t_1\} \subset \mathbf{D}^0_{2\Delta}(\mathbf{x}, t) \subset \mathbf{D}. \tag{3.13}$$

Letting k = 1 in equation (3.6), we obtain

$$\mathbf{x}^1(t) = \mathbf{z}^1(t) + \varepsilon \int_{t_0}^{t_1} \mathbf{N}(t, \sigma)\Psi(\mathbf{x}^1(\sigma), \sigma)\, d\sigma, \quad t_0 \leqslant t \leqslant t_1, \tag{3.14}$$

where

$$\mathbf{z}^1(t) = \mathbf{s}(t) + \mathbf{W}(t)\left\{\gamma - \varepsilon \mathbf{W}^{-1}(t_1) \int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma)\Psi(\mathbf{x}^0(\sigma), \sigma)\, d\sigma\right\}, \quad t_0 \leqslant t \leqslant t_1. \tag{3.15}$$

We have, on the basis of equations (3.8) and (3.15),

$$|\mathbf{z}^1(t) - \mathbf{z}^0(t)| \leqslant \varepsilon MK(t_1 - t_0) < \Delta, \qquad t_0 \leqslant t \leqslant t_1, \tag{3.16}$$

whence it follows that /171

$$\mathbf{z}^1(t) \in \mathbf{D}^0{}_\Delta(\mathbf{x}, t) = \{|\mathbf{x} - \mathbf{z}^0(t)| \leqslant \Delta,\ t_0 \leqslant t \leqslant t_1\} \subset \mathbf{D}^0{}_{2\Delta}(\mathbf{x}, t) \subset \mathbf{D}. \tag{3.17}$$

Then, as mentioned above, there exists for equation (3.14) a unique solution, defined and continuous on the interval $t_0 \leqslant t \leqslant t_1$, fitting entirely within the tube $\mathbf{D}_\Delta^1(\mathbf{x}, t)$, i.e.,

$$\mathbf{x}^1(t) \in \mathbf{D}^1{}_\Delta(\mathbf{x}, t) = \{|\mathbf{x} - \mathbf{z}^1(t)| \leqslant \Delta,\ t_0 \leqslant t \leqslant t_1\}. \tag{3.18}$$

Bearing in mind equations (3.16) and (3.18) and relying on the following inequalities for the normal forms:

$$|\mathbf{x}^1(t) - \mathbf{z}^0(t)| \leqslant |\mathbf{x}^1(t) - \mathbf{z}^1(t)| + |\mathbf{z}^1(t) - \mathbf{z}^0(t)|,$$

we obtain

$$|\mathbf{x}^1(t) - \mathbf{z}^0(t)| \leqslant 2\Delta, \qquad t_0 \leqslant t \leqslant t_1, \tag{3.19}$$

whence it follows that

$$\mathbf{x}^1(t) \in \mathbf{D}^0{}_{2\Delta}(\mathbf{x}, t) \subset \mathbf{D}. \tag{3.20}$$

Applying the method of mathematical induction to equation (3.6), it is readily shown that all approximations will fit entirely within the tube $\mathbf{D}_{2\Delta}^0(\mathbf{x}, t) \subset \mathbf{D}$.

Subtracting (3.4) from (3.14), we obtain

$$|\mathbf{x}^1(t) - \mathbf{x}^0(t)| = \Big| -\varepsilon \mathbf{W}(t)\mathbf{W}^{-1}(t_1)\int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma)\Psi(\mathbf{x}^0(\sigma), \sigma)\,d\sigma +$$

$$+ \varepsilon\int_{t_0}^{t} \mathbf{N}(t, \sigma)[\Psi(\mathbf{x}^1(\sigma), \sigma - \Psi(\mathbf{x}^0(\sigma), \sigma)]d\sigma \Big| \leqslant \Delta + \varepsilon Nr\int_{t_0}^{t} |\mathbf{x}^1(\sigma) - \mathbf{x}^0(\sigma)|\,d\sigma,\ t_0 \leqslant t \leqslant t_1$$

or

$$|\mathbf{x}^1(t) - \mathbf{x}^0(t)| \leqslant \Delta \exp[\varepsilon Nr(t_1 - t_0)], \qquad t_0 \leqslant t \leqslant t_1. \tag{3.21}$$

Setting k = 2 in equation (3.6) and subtracting (3.14) from (3.6), we obtain

$$|\mathbf{x}^2(t) - \mathbf{x}^1(t)| = \Big|\varepsilon \mathbf{W}(t)\mathbf{W}^{-1}(t_1)\int_{t_0}^{t_1} \mathbf{N}(t_1, \sigma)[\Psi(\mathbf{x}^0(\sigma)\sigma) - \Psi(\mathbf{x}^1(\sigma), \sigma)]d\sigma +$$

$$+ \varepsilon\int_{t_0}^{t} \mathbf{N}(t, \sigma)[\Psi(\mathbf{x}^2(\sigma), \sigma) - \Psi(\mathbf{x}^1(\sigma), \sigma)]d\sigma\Big| \leqslant \Delta \exp[\varepsilon Nr(t_1 - t_0)]\varepsilon Mr(t_1 - t_0) +$$

$$+ \varepsilon Nr\int_{t_0}^{t} |\mathbf{x}^2(\sigma) - \mathbf{x}^1(\sigma)|\, d\sigma, \qquad t_0 \leqslant t \leqslant t_1$$

or

$$|\mathbf{x}^2(t) - \mathbf{x}^1(t)| \leqslant \Delta \exp[\varepsilon Nr(t_1 - t_0)]\{\varepsilon Mr(t_1 - t_0)\exp[\varepsilon Nr(t_1 - t_0)]\}, \quad t_0 \leqslant t \leqslant t_1. \tag{3.22}$$

Applying mathematical induction, the validity of the following estimations is easily established:

$$|\mathbf{x}^k(t) - \mathbf{x}^{k-1}(t)| \leqslant \Delta \exp[\varepsilon Nr(t_1 - t_0)]\{\varepsilon Mr(t_1 - t_0)\exp[\varepsilon Nr(t_1 - t_0)]\}^{k-1}, \quad t_0 \leqslant t \leqslant t_1 \quad (k = 1, 2, \ldots). \tag{3.23}$$

Denoting

$$Q(\varepsilon) = \varepsilon Mr(t_1 - t_0)\exp[\varepsilon Nr(t_1 - t_0)], \tag{3.24}$$

we conclude that the series

$$\Delta \exp[\varepsilon Nr(t_1 - t_0)] + \Delta \exp[\varepsilon Nr(t_1 - t_0)]Q(\varepsilon) + \Delta \exp[\varepsilon Nr(t_1 - t_0)]Q^2(\varepsilon) + \ldots \tag{3.25}$$

converges (by virtue of the D'Alembert test) under the stipulation that

$$0 < Q(\varepsilon) < 1. \tag{3.26}$$

Inasmuch as all terms of the series

$$[\mathbf{x}^1(t) - \mathbf{x}^0(t)] + [\mathbf{x}^2(t) - \mathbf{x}^1(t)] + [x^3(t) - \mathbf{x}^2(t)] + \ldots \tag{3.27}$$

are less in normal form than the corresponding terms of the series (3.25), the /172 series (3.27) not only converges, it converges uniformly for all $t \in [t_0, t_1]$ by virtue of the Weierstrass criterion.

Because each term of the series (3.27) is a continuous vector time function, the limit of the sequence $\{x^k(t)\}$ exists and is a continuous vector time function:

$$\lim_{k\to\infty} x^k(t) = x(t) \in D^0_{2\Delta}(x,\ t) \subset D. \tag{3.28}$$

Bearing in mind equations (3.5) and (3.6) and passing to the limits as $k \to \infty$, it is readily seen that the vector function $x(t)$ satisfies the matrix integral equation (3.2), and

$$\lim_{k\to\infty} q^{k+1} = q \tag{3.29}$$

satisfies the relation (3.1).

It still remains for us to explain at what values of ϵ the condition (3.26) will be fulfilled.

A graph of the function $Q(\epsilon)$ is illustrated by the heavy curve in figure 2. It is easily shown that the curve of $Q(\epsilon)$ is tangent to the straight line

$$Q_1 = Mr(t_1 - t_0)\varepsilon \tag{3.30}$$

at the origin.

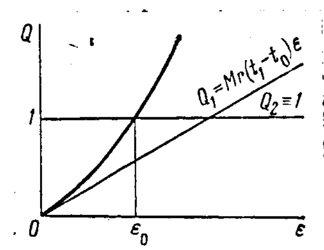

Figure 2

Then the curve of $Q(\epsilon)$ necessarily intersects the straight line $Q_2 \equiv 1$ at some point $\epsilon = \epsilon_0$. The value of ϵ_0 is that value which figures in the condition (3.10).

Thus, under the conditions 1), 2), and 3), the convergence of the successive approximations has been proven.

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